STATISTICAL MODEL OF ANISOTROPIC, FIBROUS, POROUS BODIES

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In [1] note is made of the need for a clear separation of two problems of the physics of porous systems: description of the porous medium itself and consideration of the processes taking place in this medium. For example, the motion of a liquid or the motion of Brownian particles in the pore space of a porous medium cannot be studied successfully if a correct description of the medium itself is not available. We give below a description of anisotropic porous bodies of fiberglas, a material widely used as filtrational and heat insulating media. We give constructive definitions of two concepts, previously used in an intuitive sense: the "structure of a porous body" and "a homogeneous (nonhomogeneous) porous body." For a random medium, realizations of which are model porous bodies with a given structure, we calculate typical averaged parameters and their distributions, used in transmission and mechanical properties calculations. We investigate a method for experimentally analyzing the structure of real nonhomogeneous porous bodies, based on a study of the properties of the random dependence of the porosity coefficient on the coordinates.

1. Concepts of "Structure" and "Homogeneity" as Applied to Porous Bodies. We consider the structure of a porous body as being given if the following two features are known: a) the geometrical figures and relative sizes of the initial structural elements, for example, the fibers, and b) the probability law defining the way in which the initial structural elements are jointly distributed in space.

Let x denote some property of a porous body. The concept of a "porous body homogeneous (nonhomogeneous) with respect to x" is defined in terms of the concept of structure as follows: a porous body is said to be homogeneous (nonhomogeneous) with respect to x if the structure features of the body, which have an influence on the parameter chosen, do not (do) depend on the coordinate. If we are dealing with porous bodies with various structures, it is more convenient to use formulations of the type of a "porous body with a structure y homogeneous (nonhomogeneous) with respect to x." In this case we regard the structure of the porous body as being homogeneous (nonhomogeneous) with respect to x if the structural features influencing x do not (do) depend on the coordinate.

2. Model Porous Bodies with a Structure A Homogeneous with Respect to α. Suppose that we are given a random medium in the form of the pair $\{\Lambda(A), dP(\lambda)\}$, where $\Lambda(A)$ is a set of porous bodies λ, described in detail, with a structure A homogeneous with respect to α, while $dP(\lambda)/d\lambda$ is the probability density of each realization of $\Lambda(A)$.

Let $Q_0^{(i)}$ denote the set of points of a certain i-th rectilinear fiber. We distinguish a cylindrical system of coordinates $\{\rho, \varphi, z\}$, with the z axis directed towards the unit vector **k**. Let $\mathbf{r}_i=\{\rho_i, \varphi_i, z_i\}$ be the point of intersection of the axis of the i-th fiber $Q_0^{(i)}$ with the plane Q containing the z axis and perpendicular to the projection of the axis of the fiber on the plane $z=0$. Let ω_i be the angle between the z axis and the axis of the fiber. The coordinates $\mathbf{r}_i$, ω_i uniquely define the position of the i-th infinite fiber in space.

We define $\Lambda(A)$ as the set of all possible model porous bodies (filters) with a structure A homogeneous with respect to α, satisfying the following conditions:

1) Each filter belonging to $\Lambda(A)$ has equal geometrical figures G, oriented identically to the vector **k**; in particular, let G be the right circular cylinder bounded by the planes $z=0$, $z=h$ and the cylindrical surface $\rho=R$.

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2) In accord with the structural condition, all the filters of $\Lambda(A)$ are formed by straight circular fibers of radius a, with no termini inside G, where the fibers are positioned independently of one another and their coordinates $\rho_i, \varphi_i, z_i, \omega_i, a_i, (i = 1, 2, \ldots, m; m = 0, 1, 2, \ldots)$ are independent random variables with the density distributions

$$R^{-1}, (2\pi)^{-1}, h^{-1}, \delta(\omega_i - \pi 2^{-1}), f(a) \tag{2.1}$$

respectively.

3) The average number of fibers intersecting the domain G is equal to $\langle m \rangle$.

4) We assume that

$$\langle a \rangle R^{-1} \ll 1, \ \langle a \rangle h^{-1} \ll 1 \tag{2.2}$$

The density distributions (2.1) correspond approximately to those for real anisotropic fiberglas filters and, consequently, the given set $\Lambda(A)$ of model porous bodies is an approximate representation of the corresponding set of real anisotropic fibered porous bodies with the appropriate structure and with the characteristics G, $f(a)$, and $\langle m \rangle$.

From the distributions (2.1) it follows that the function $dP(\lambda)/d\lambda$, which gives the probability density for finding a filter $\lambda = (a_1, a_2, \ldots, a_m; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m)$ consisting of m rectilinear fibers, having radii a_i and coordinates $\mathbf{r}_i$, has the form

$$dP(\lambda) = \frac{1}{m!}\left(\frac{\langle m \rangle}{2\pi R h}\right)^m \exp(-\langle m \rangle) \prod_m f(a_i)\, da_i\, d\rho_i\, d\varphi_i\, dz_i \tag{2.3}$$

$$\rho \in [0, R], \ \varphi \in [0, 2\pi], \ z \in [0, h]$$

The technique for deriving Eq. (2.3) is analogous, for example, to that used in [2].

According to Eq. (2.3), the probability p(m) that an arbitrary filter of $\Lambda(A)$ consists of m fibers is equal to

$$p(m) = \langle m \rangle^m \exp(-\langle m \rangle) / m! \tag{2.4}$$

wherein, for large values of $\langle m \rangle$, we have actually [3]

$$\operatorname{Prob}\{m \leqslant x \mid \langle m \rangle\} \approx \Phi((x + 0.5 - \langle m \rangle)\langle m \rangle^{-1/2}) \tag{2.5}$$

$$\Phi(y) = (2\pi)^{-1/2} \int_{-\infty}^{y} \exp(-t^2/2)\, dt$$

Let $\Lambda_1(A)$ be a subset of $\Lambda(A)$ consisting of those filters in which the point $\mathbf{r}$ does not belong to the fibers. Using the equation

$$\langle \beta \rangle = \int_{\Lambda_1(A)} dP(\lambda)$$

we obtain a relationship between $\langle m \rangle$ and the other parameters of the filter:

$$\gamma \langle m \rangle = V\langle \alpha \rangle / \langle \zeta \rangle \pi \langle a^2 \rangle$$

$$\gamma = -\langle \alpha \rangle / \ln \langle \beta \rangle, \ \langle \zeta \rangle = 2^{-1}\pi R, \ \langle \beta \rangle \leqslant 1 \tag{2.6}$$

where V is the volume of the domain G, ζ_i is the length of the segment of the i-th fiber included in G, and $\langle \alpha \rangle = 1 - \langle \beta \rangle$.

From Eqs. (2.6) we have $\langle \alpha \rangle \to 1$ for $\langle m \rangle \to \infty$. This is the result of the fact that in the model filter the fibers, in meeting, penetrate one another rather than bend around each other. We can use the quantity $\gamma \in [0, 1]$ as one of the criteria for the correspondence of the model to real filters, assuming that for $\gamma = 1$ there is complete correspondence with respect to γ.

Let P_j' be the probability that some point in the space of an arbitrary filter of $\Lambda(A)$ belongs simultaneously to j fibers (j = 0, 1, 2, ...). Using Eqs. (2.3) and (2.6), we obtain

$$P_j' = (-1)^j \langle \beta \rangle \ln^j \langle \beta \rangle / j! \tag{2.7}$$

For small $\langle\alpha\rangle$ we have the inequality $P_1' \gg \sum_{j=2} P_j'$ and, consequently, $\gamma \approx 1$. In particular, the widely used fiberglas filters and the FP filters are characterized by small $\langle\alpha\rangle$.

From Eq. (2.3) we find that the probability that a spherical particle will traverse a free rectilinear path of distance l not less than x in the direction of the unit vector $\mathbf{\Omega}$ in an arbitrary filter of Λ(A) is given by

$$\operatorname{Prob}\{l \geqslant x \mid \Omega, r\} = \langle\beta\rangle^{\nu} \exp\left[-x\langle\beta\rangle^{\nu} / \langle l(\Omega, r)\rangle\right] \tag{2.8}$$

where

$$\langle l(\Omega, r)\rangle = -\frac{\pi}{2}\frac{\langle(a+r)^2\rangle}{\langle a+r\rangle}\frac{\langle\beta\rangle^{\nu}}{\nu\ln\langle\beta\rangle}\frac{\pi}{2\mathrm{E}(|\Omega\times\mathbf{k}|)} \tag{2.9}$$

is the mean free rectilinear path of the spherical particle of radius r in the direction of the vector $\mathbf{\Omega}$ in an arbitrary filter of Λ(A), **E** is the complete elliptic integral of the second kind, and $\nu = \langle(a+r)^2\rangle / \langle a^2\rangle$.

The term $\langle\beta\rangle^{\nu}$ before the exponential term in Eq. (2.8) takes into account the probability that in the initial instant the path of the particle is free.

For $|\Omega' \times \mathbf{k}| = 1$ and $\langle\alpha\rangle \ll 1$ the expression (2.9) may be written in the form

$$\langle L_0\rangle = \pi\langle\beta\rangle / 2\langle l(\Omega', 0)\rangle \tag{2.10}$$

where $\langle L_0\rangle$ is the average total length of the fibers encountered per unit area of the surface of a filter of thickness $h = 2\langle a\rangle$, and $\langle l(\Omega', 0)\rangle\langle\beta\rangle^{-1}$ is the mean free path in the plane populated with infinite straight lines according to the law (2.1) with mean concentration $\langle L_0\rangle$. The expression (2.10) coincides with the result given in [4].

In accordance with Eq. (2.8), the exponent in expression (7) in [5] should be corrected by multiplying it by the quantity $\langle\beta\rangle^{\nu}$. Also in [5], the expression (3) for the mean free path must be corrected in accordance with the result (2.9).

We now take up the problem concerning the intersections of fibers in filters. The ways of defining an N-fold intersection of fibers are not unique for N > 2. To calculate the concentration of intersections of fibers with the help of Eq. (2.9), one must assume that the fiber $Q_0^{(1)}$ takes part in an N-fold intersection if $\bigcup_{j=2}^{N} z(1, j)$ is a connected domain, where z(i, j) is the projection of $Q_0^{(i)} \cap Q_0^{j}$ on the plane z = 0. In the calculation per unit filter volume, let the quantity n_i denote the average number of i-fold fiber intersections. Using the Eqs. (2.6) and (2.9), we obtain the asymptotic expression

$$2n_2 + 3n_3 + 0(n_4) = 8\langle a\rangle \ln^2\langle\beta\rangle / \pi^3\langle a^2\rangle^2 \tag{2.11}$$

in which the coefficient 3 of n_3 is only proper when $n_4 \ll n_3$. According to the estimate (2.7), when $\langle\alpha\rangle \ll 1$, actually $n_3 \ll n_2$, and the expression (2.11) gives a concentration for two-fold fiber intersections which is $2/\pi$ times less than the result given in [6] and $2/3$ times less than the result given in [7].

From Eqs. (2.7) and (2.11) it follows automatically that the mean intersectional volume v' when two fibers in filters of Λ(A) intersect is equal to

$$\lim_{R\to\infty} v' = \lim_{\langle\alpha\rangle\to 0} P_2' / n_2 = \pi^3\langle a^2\rangle^2 / 8\langle a\rangle \tag{2.12}$$

The expressions (2.11) and (2.12) are useful for studying the properties of filters such as strength, elasticity, thermal conductivity, etc.

We consider an important characteristic of the random medium $\{\Lambda(A), dP(\lambda)\}$, namely, the function $H_x(\alpha)$, which is the probability density for the event that an arbitrary realization of Λ(A) with a given geometry G has a space occupancy coefficient equal to α. Suppose that m fibers intersect G according to the rule (2.1). Let $\tau_m(\alpha)$ denote the density of the probability that the space occupancy coefficient in G is equal to α. If we put $\gamma = 1$, then τ_m is obtained by Markov's method [8]:

$$\begin{aligned}\tau_m(\alpha)\,d\alpha = \frac{d\alpha}{\sqrt{2\pi}\,\sigma_m}\exp\left(-\frac{z^2}{2}\right)\Big\{1 - \frac{z}{2m^{1/2}}\frac{\varkappa_3}{\varkappa_2^{3/2}}\left[1 - \frac{2}{3}\left(\frac{z^2}{2}\right)\right] + \\ + \frac{1}{8m}\frac{\varkappa_4}{\varkappa_2^2}\left[1 - 4\left(\frac{z^2}{2}\right) + \frac{4}{3}\left(\frac{z^2}{2}\right)^2\right] + 0(zm^{-3/2})\Big\}\end{aligned} \tag{2.13}$$

$$z = (\alpha - m\langle\xi\rangle)\,\sigma_m^{-1}, \quad \sigma_m^2 = m\varkappa_2$$

where $\varkappa_i$ are the semiinvariants [9] of the composite random variable $\xi = \zeta\pi a^2 V^{-1}$. For a filter with geometry G we have $\langle\zeta^n\rangle = 4R^2 \langle\zeta^{n-2}\rangle n(n+1)^{-1}$. Finally, using the results (2.4) and (2.13), we obtain the following expression for the desired function H_x:

$$H_x(\alpha) = \sum_{m=0}^{\infty} p(m)\,\tau_m(\alpha) \tag{2.14}$$

In accordance with a lemma concerning the limit of a composite random function [10], the asymptotic expression for Eq. (2.14) has the form

$$H_x(\alpha) = (2\pi\sigma_x^2)^{-1/2} \exp\left[-(\alpha - \langle\alpha\rangle)^2 / 2\sigma_x^2\right]$$
$$\sigma_x^2 = \langle m\rangle\langle\xi^2\rangle = 16\langle\alpha\rangle\langle a^4\rangle / 3\pi Rh\langle a^2\rangle, \quad \langle m\rangle \gg 1 \tag{2.15}$$

It should be emphasized that for fixed $\langle\alpha\rangle$ and V the parameter σ_x depends, through $\langle m\rangle$ and $\langle\zeta^2\rangle$, on the geometry of the figure G and on its orientation to the vector **k**.

The method used above for determining the free path in filters of Λ(A) may be easily generalized to the case of filters of noncircular fibers. We consider an infinite model filter of ribbon fibers with rounded-off edges and with the wide side oriented perpendicular to the vector **k**. The calculation for such filters of the mean free path of a thin beam is made with a modified formula of the type (2.3), giving

$$\langle l(\Omega, 0)\rangle = -s\langle\beta\rangle\{2b'[\chi(\Omega\cdot\mathbf{k}) + 2\pi^{-1}\mathrm{E}(|\Omega\times\mathbf{k}|)]\ln\langle\beta\rangle\}^{-1} \tag{2.16}$$
$$b'\chi = a' - b', \quad s = \pi b'^2 \quad (1 + 4\pi^{-1}\chi)$$

where $2a'$ is the fiber width, $2b'$ is the fiber thickness, and s is the cross-sectional area of the ribbon fiber.

We assume that for constant s and $\langle\alpha\rangle$ the thickness of the ribbon fibers in the filters decreases indefinitely. Then

$$\lim_{\chi\to\infty}\langle l(\Omega, 0)\rangle = \begin{cases} 0, & (\Omega\cdot\mathbf{k}) \neq 0 \\ \infty, & (\Omega\cdot\mathbf{k}) = 0 \end{cases} \tag{2.17}$$

The first limit in Eq. (2.17) is real only for filters extending to infinity in a plane perpendicular to **k**.

The expressions (2.9) and (2.16) show that the free path in a porous body is a function of the geometrical figure of the initial structural elements and their orientation in space.

3. Ejections of a Random Process $\alpha(t)$ for Porous Bodies with a Structure A Homogeneous with Respect to α. For a porous body with structure A homogeneous with respect to α and bounded by the infinite planes $z=0$ and $z=h$, let the random function $\alpha(t)$, $t\in(-\infty, \infty)$ define the space occupancy coefficient in a cylindrical region G as a function of the coordinate $t=\rho/2R$ for $\varphi=\mathrm{const}$, where (ρ, φ) is the point of intersection of the axis of the cylinder G with the plane $z=0$. According to Eq. (2.3), this function is ergodic [11] and independent of φ. According to Eq. (2.15) with $\langle m\rangle \gg 1$, the function $\alpha(t)$ is also a normal function. We find the mean number of ejections of the graph of the function $\alpha(t)$ at the level $\langle\alpha\rangle$ per unit interval of the t axis.

We write the function $\alpha(t)$ in the form

$$\alpha(t) = X(t, \langle\alpha\rangle) + \langle\alpha\rangle, \quad \langle X\rangle = 0 \tag{3.1}$$

The correlation function $K_x(\tau)$ of the process $\alpha(t)$ has the form

$$K_x(\tau) = \langle X_1 X_2\rangle = \sigma_x^2 \frac{2}{\pi}\int_{\theta_0}^{\pi/2} I(k)\,d\theta \tag{3.2}$$

where

$$X_i = X(t_i, \langle\alpha\rangle), \quad k^2 = 1 - \tau^2\cos^2\theta, \quad \tau = (\rho_1 - \rho_2)/2R$$
$$I(k) = (2 - k^2)\,\mathrm{E}(k) - 2(1 - k^2)\,\mathrm{K}(k), \quad \theta_0 = \begin{cases} \arccos|1/\tau|, & |\tau| \geqslant 1 \\ 0, & |\tau| \leqslant 1 \end{cases}$$

K is the complete elliptic integral of the first kind, and σ_x is defined in Eq. (2.15).

Calculation of the expression (3.2) gives

$$k_x(\tau) = \sum_{m=0}^{\infty}(a_m + b_m \ln|\tau|)\,|\tau^m|, \quad |\tau| \leqslant 1 \tag{3.3}$$

$$k_x(\tau) = \sum_{m=0}^{\infty} c_m |\tau^{-m}|, \quad |\tau| \geqslant 1 \tag{3.4}$$

where $k_x(\tau)$ is the correlation coefficient and the values of the coefficients a_m, b_m, c_m are given in Table 1.

The process $\alpha(t)$ gives an infinite concentration of ejecta since its correlation function does not have a second derivative at the point $\tau = 0$. An actual apparatus with a finite response registers separately the intersections of the graph of $\alpha(t)$ with the level $\langle\alpha\rangle$ at points t_1 and t_2, providing that $|t_1 - t_2| \geq \Delta$, where Δ is a constant of the apparatus. Determination of the concentration of such "crude" ejecta for multidimensional Markov processes is reduced to the solution of a differential equation of the type of Kolmogorov's second equation [11]. For the process in question this method is involved; therefore it is desirable to obtain a result within the scope of correlation theory.

Let us suppose that we can neglect the fine details of the graph of the function $\alpha(t)$ on intervals of the t axis less than Δ. We smooth the graph of the function in the domain $t \in [-n\Delta, n\Delta]$, $n = 1, 2, 3, \ldots$ as follows:

1) On the t axis we isolate the points $t_i = (i-n)\Delta$, $i = 0, 1, 2, \ldots, 2n$.
2) Let $L_{\Delta,n}$ denote the linear homogeneous operation of constructing the Lagrange interpolating polynominal of degree 2n, which coincides with the values of $\alpha(t)$ at the points t_i.

Then

$$L_{\Delta,n}[\alpha(t)] = \alpha_{\Delta,n}(t) \tag{3.5}$$

The correlation coefficient $k_{x,\Delta,n}(\tau)$, $\tau \in [-n\Delta, n\Delta]$ for the random function $\alpha_{\Delta,n}(t)$ on a bounded interval has the form

$$k_{x,\Delta,n}(\tau) = L_{\Delta,n} L_{\Delta,n}[k_x(\tau)] \tag{3.6}$$

For the entire axis we have, respectively,

$$k_{x,\Delta}(\tau) = \lim_{n\to\infty} k_{x,\Delta,n}(\tau) \tag{3.7}$$

The second and fourth derivatives of $k_{x,\Delta}$ at the point $\tau = 0$ are equal to

$$k''_{x,\Delta}(0) = \lim_{n\to\infty} \frac{2}{\Delta^2} \sum_{i=1}^{n} d_i(n)[1 - k_x(i\Delta)] \tag{3.8}$$

$$k^{(4)}_{x,\Delta}(0) = -\lim_{n\to\infty} \frac{24}{\Delta^4} \sum_{i=1}^{n} d_i(n) D_i(n)[1 - k_x(i\Delta)]$$

where

$$d_i(n) = (-1)^i \frac{2}{i^2} \frac{n!\,n!}{(n-i)!\,(n+i)!}, \quad D_i(n) = \sum_{j=1}^{n} \frac{1}{j^2} - \frac{1}{i^2}$$

If the expansion of $k_x(\tau)$ in a series of the type (3.3) is, in fact, for the entire τ axis, then substituting Eq. (3.3) into Eq. (3.8), we obtain

$$k''_{x,\Delta}(0) = 2 \sum_{m=1}^{\infty} (s_{1,m} a_m + s_{2,m} b_m + s_{1,m} b_m \ln\Delta)\,\Delta^{m-2} \tag{3.9}$$

$$k^{(4)}_{x,\Delta}(0) = 24 \sum_{m=1}^{\infty} (s_{3,m} a_m + s_{4,m} b_m + s_{3,m} b_m \ln\Delta)\,\Delta^{m-4}$$

where

$$s_{1,m}(n) = -\sum_{i=1}^{n} d_i(n)\, i^m, \quad s_{2,m}(n) = -\sum_{i=1}^{n} d_i(n)\, i^m \ln i$$

$$s_{3,m}(n) = \sum_{i=1}^{n} d_i(n) D_i(n)\, i^m, \quad s_{4,m}(n) = \sum_{i=1}^{n} d_i(n) D_i(n)\, i^m \ln i$$

$$s_{j,m} = \lim_{n\to\infty} s_{j,m}(n)$$

The values of the coefficients $s_{j,m}$ are presented in Table 2.

TABLE 1

n	a_{2n}	b_{2n}	c_{2n+1}
0	1.000 000	0.000 000	0.292 180
1	—0.809 581	0.750 000	0.018 458
2	0.122 773	—0.070 312	0.004 341
3	0.005 830	0.007 324	0.001 596
4	0.001 074	0.002 002	—
5	0.000 321	0.000 788	—

TABLE 2

j	$s_{j,1}$	$s_{j,2}$	$s_{j,3}$	$s_{j,4}$	$s_{j,5}$
1	1.3862	1.0000	0.5000	0.0000	—0.2516
2	—0.3196	—0.4514	—0.5304	—0.4262	—0.0312
3	—0.4774	0.0000	0.5637	1.0000	0.9138
4	0.4063	0.5401	0.5529	0.2497	—0.4790

As a consequence of the fact that the series of the form (3.3), in the case considered, is real only for $|\tau| \leq 1$, the error in calculating the derivatives of $k_{x,\Delta}$ according to the formulas (3.9) increases with an increase in Δ, amounting to less than a percent for $\Delta \leq 0.5$ and increasing to 5% for $\Delta = 1$.

In the calculation per unit interval of τ the average number of ejecta $N_{x,\Delta}(\tau_0)$ of the random differentiable function $\alpha_\Delta(t)$ at the level $\langle\alpha\rangle$, the duration of which exceeds τ_0, is equal to

$$N_{x,\Delta}(\tau_0) = N_{x,\Delta}(0)\left(1 - \int_0^{\tau_0} p(\tau)\,d\tau\right) \tag{3.10}$$

where $p(\tau)$ is the density of the probability that an ejection has the duration τ.

According to [12], for differentiable processes,

$$N_{x,\Delta}(0) = \pi^{-1}\sqrt{-k''_{x,\Delta}(0)} \tag{3.11}$$

$$p(\tau) \leqslant p_0(\tau) = -\tau\left(k^{(4)}_{x,\Delta}(0) - k''^{2}_{x,\Delta}(0)\right)/8k''_{x,\Delta}(0) \tag{3.12}$$

where in Eq. (3.12) the equality sign applies only for $\tau \to 0$. The derivatives of $k_{x,\Delta}(\tau)$ are defined in Eqs. (3.8) and (3.9). As is evident from Eq. (3.10), the function $N_{x,\Delta}(\tau_0)$ does not depend on $\langle\alpha\rangle$, R, and on $f(a)$. Only the amplitude of the function $\alpha_\Delta(t)$ depends on these parameters. The concentration of the ejecta $N'_{x,\Delta}(\tau_0)$ in the calculation per unit length of the ρ interval is equal to

$$N'_{x,\Delta}(\tau_0) = N_{x,\Delta}(\tau_0)/2R \tag{3.13}$$

This means that

$$N'_{x,\Delta}(\tau_0)R = \text{const} \tag{3.14}$$

where the quantity R is restricted by the condition (2.2).

It is of interest to compare the theoretical expressions (3.9), (3.10)-(3.12) with experimental results. On an MF-4 microphotometer we registered the function i(t) of an optically transparent real fiberglas filter with a suitable structure and with parameters $\langle\alpha\rangle = 8.52\cdot10^{-3}$, $\langle a\rangle = 1.57\cdot10^{-4}$ cm, $\langle a^2\rangle = 2.87\cdot10^{-8}$ cm^2; h = 0.316 cm. We assumed that the points of intersection by the graph of the function i(t) of level $\langle i\rangle$ correspond to the points of intersection by the graph of the function $\alpha(t)$ of level $\langle\alpha\rangle$. We used the order of calculating the intersections, partially excluding the influence of the nonhomogeneities of the real filter on the result. For the quantity α_0 satisfying the condition

$$0.99 = \int_{\langle\alpha\rangle-\alpha_0}^{\langle\alpha\rangle+\alpha_0} H_x(\alpha)\,d\alpha$$

we defined the corresponding quantity i_0 and assumed that in the filter homogeneity was violated in that region of t values where the graph of i(t) goes beyond the limits of the region $[\langle i\rangle - i_0, \langle i\rangle + i_0]$. In calculating the ejecta of the function i(t) we discarded ejecta with a duration less than 0.1 cm on the diagrammatic ribbon and also discarded ejecta with amplitude $|i - \langle i\rangle| > i_0$. The experimental values obtained for the concentration of ejecta were 0.49 and 0.41, respectively, for values of $\Delta = 0.277$ and 0.555. The theoretical values of $N_{x,\Delta}(\Delta)$ corresponding to these experimental values are approximately 0.57 and 0.48.

4. Model of a Filter with a Structure A Nonhomogeneous with Respect to α. We consider a model of a filter with a structure A nonhomogeneous with respect to α in the form

$$\alpha(t) = X(t, \alpha_1) + \alpha_1 \qquad (4.1)$$
$$\alpha_1 = Y(t) + \langle\alpha\rangle, \ \langle X\rangle = 0, \ \langle Y\rangle = 0$$

By analogy with Eq. (3.1) the random function α_1 in Eq. (4.1) plays the role of a local mean value, where Y varies more slowly in comparison with X for a given R. This means that in Eq. (2.3) the parameter $\langle m\rangle$ is replaced by a function of the coordinate which varies little at distances of order 2R (the second structural feature). Using the representation (4.1), we reduce the study of the structure of a nonhomogeneous filter to the study of the function Y(t).

The correlation function of the process (4.1) has the form

$$K_\alpha(\tau) = \langle X_1X_2\rangle + \langle X_1Y_2\rangle + \langle Y_1X_2\rangle + \langle Y_1Y_2\rangle \qquad (4.2)$$
$$X_i = X(t_i, \ Y(t_i) + \langle\alpha\rangle), \ Y_i = Y(t_i)$$

Assuming that a section of the random function α_1 has a normal distribution density $H_y(\alpha_1)$ with parameters σ_y and $\langle\alpha\rangle$, we obtain

$$K_\alpha(\tau) = \langle X_1X_2\rangle + \langle Y_1Y_2\rangle \qquad (4.3)$$

where the function $\langle X_1X_2\rangle$ is defined in Eqs. (3.2)-(3.4). Thus the correlation function of the process $\alpha_1(t)$ is equal to

$$K_y(\tau) = K_\alpha(\tau) - K_x(\tau) \qquad (4.4)$$

and, consequently,

$$\sigma_y^2 = K_\alpha(0) - \sigma_x^2 \qquad (4.5)$$

where $K_\alpha(\tau)$ is determined experimentally.

The parameters $\langle\alpha\rangle$, σ_y and the function $K_y(\tau)$ furnish fairly complete information concerning the quality of a nonhomogeneous filter. Instead of K_y we can use the concentration N_y of ejecta of the function α_1 at the level $\langle\alpha\rangle$, referred to a unit of the interval $\rho/2R$. The quantity giving the mean duration of ejecta, namely, $L_y(R) = 2R/N_y$, we regard as an averaged scale of the nonhomogeneities (jointly with σ_y^2). General considerations show that $L_y(R)$ is constant for $2R \le L_x'$ and increases for $2R > L_y'$, where L_y' is the duration of the shortest ejection of the function α_1. For homogeneous filters

$$\sigma_y = 0, \ N_y = 0, \ K_\alpha = K_x, \ L_y(R) = \infty$$

The analysis given above has proceeded on the assumption that in nonhomogeneous filters the density of the distribution h^{-1} is constant in the k direction (2.1). If this assumption is not made, we must then measure yet another scale of the nonhomogeneities in the k direction.

For an n-fold ($n \ge 0$) linear compression of nonhomogeneous filters in the k direction the volume of a region of the filter with a "space occupancy coefficient" α_1 changes and becomes equal to $n\alpha_1$. If the random quantity α_1 has a normal probability density, then we have, in fact, the following expression for the filter after compression:

$$H_y(n\alpha_1)\, d(n\alpha_1) = \frac{d(n\alpha_1)}{\sqrt{2\pi}\, n\sigma_y} \exp\left[-\frac{(n\alpha_1 - n\langle\alpha\rangle)^2}{2n^2\sigma_y^2}\right] \qquad (4.6)$$

Thus when the filter is compressed, the dispersion $n^2\sigma_y^2$ varies in proportion to the square of the degree of compression.

We consider now a possible way of determining σ_y, using hydrodynamic properties of filters. Let $L_y(R)$, $R \le L_y'$ be the mean length of an ejection of the function $\alpha_1(t)$. Then a fine filter, nonhomogeneous with respect to α, with a surface much larger than L_y^2 must present a resistance Δp to the flow of a gas, where Δp is equal to

$$\Delta p = \frac{4\mu h \langle u\rangle \langle\alpha\rangle}{\langle a^2\rangle k'(\alpha\gamma', c_1', c_2')} \qquad (4.7)$$

$$k' = b'c_1' \ln(\langle\alpha\rangle \gamma') + c_2' + a_0'c_1' + \sum_{i=1}^{\infty} a_i' \langle\alpha_1^{i-1}\rangle \langle\alpha\rangle \gamma'^i$$

$$c_1' = \sum_{n=0}^{\infty} (-1)^n \frac{\langle(\alpha_1 - \langle\alpha\rangle)^n\rangle}{\langle\alpha_1\rangle^n}, \quad c_2' = b' \sum_{n=1}^{\infty} (-1)^{n+1} N_n \frac{\langle(\alpha_1 - \langle\alpha\rangle)^n\rangle}{\langle\alpha_1\rangle^n}$$

$$N_n = N_{n-1} + n^{-1}, \quad N_1 = 1$$

where μ is the viscosity of the gas, $\langle u\rangle$ is the mean speed of the flow of the gas ahead of the filter, and a_i', b', γ' are constant coefficients. By the property (4.6) the coefficients c_1', c_2' also remain unchanged during a linear compression of the filter. In Eqs. (4.7) the form of the function k' for a filter ($c_1'=1$, $c_2'=0$) homogeneous with respect to α is that given in [13].

Treating the experimentally determined dependence of Δp on $\langle\alpha\rangle$ for a real filter, nonhomogeneous with respect to α, in accordance with Eqs. (4.7), we can probably determine all the constant coefficients in the function k' and, consequently, the dispersion σ_y^2.

5. Some Other Structures. We give below three schematic examples in order to get some feel for the difference in the properties of a random function for various structures.

Structure B. Let filters with a structure B, homogeneous with respect to α, be made of rectilinear fibers perpendicular to a plane Q, where the points of intersection of the fiber axes with this plane form a two-dimensional Poisson field of points. The remaining features of the filters are the same as those described in Sec. 2. For such a structure

$$k_x(\tau) = \begin{cases} I(k); & |\tau\cos\theta| \leqslant 1 \\ 0, & |\tau\cos\theta| \geqslant 1 \end{cases} \tag{5.1}$$

where θ is the angle between the plane Q and the direction ρ, and the function I(k) is defined in Eq. (3.2).

The section of the random function $\alpha(t)$ has the density of the distribution (2.15) for $\theta \neq \pi 2^{-1}$, and $H_x(\alpha) = \delta(\alpha - \alpha')$, $k_x = 1$ for $\theta = \pi 2^{-1}$, where the "constant" quantity α' also has the density of the distribution (2.15). When $\theta = \pi 2^{-1}$, the function $\alpha(t)$ loses its ergodic property.

Structure C. Let filters with a structure C, homogeneous with respect to α, be formed from rectilinear mutually parallel fibers, where the points of intersection of the axes of these fibers with a plane perpendicular to them form the nodes of a rectangular mesh with parameters d and d'. The correlation coefficient of the process $\alpha(t)$ for such a filter has the form

$$k_x(\tau) = \begin{cases} \psi(\tau, \theta), & \theta \neq \pi/2 \\ 1, & \theta = \pi/2 \end{cases} \tag{5.2}$$

where ψ is a periodic function with the period $d/2R\cos\theta$ and, consequently, the concentration of the ejecta of the function $\alpha(t)$ above the level $\langle\alpha\rangle$ is equal to $2R\cos\theta/d$ in the calculation made per unit of the τ interval.

Structure A'. In the filters of $\Lambda(A)$, homogeneous with respect to α, let a portion of the circular fibers be replaced by aggregates of double, triple, etc.(uncarded) fibers. A structure A' is then formed which is different from the structure A, homogeneous with respect to α, since the set of initial structural elements now contains, along with the single fibers, also uncarded aggregates. If, in a first approximation, we consider such aggregates as circular fibers with increased radii, then the presence of the latter in filters manifests itself, in accordance with Eq. (2.15), in an increased [in comparison with filters of $\Lambda(A)$] amplitude of the random oscillations of the function $\alpha(t)$, without at the same time changing the concentration of its intersections with the level $\langle\alpha\rangle$. Filters with uncarded fibers are encountered in practice.

The examples given show that the analysis of the random function $\alpha(t)$ enables us to obtain useful information concerning the structure of a given fibered material.

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